

VIRTUALIZATION MAP FOR THE LITTELMANN PATH MODEL

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ABSTRACT. We show the natural embedding of weight lattices from a diagram folding is a virtualization map for the Littelmann path model, which recovers a result of Kashiwara. As an application, we give a type independent proof that certain Kirillov–Reshetikhin crystals respect diagram foldings, which is a known result on a special case of a conjecture given by Okado, Schilling, and Shimozono.

1. INTRODUCTION

In [Kas96], Kashiwara gave a construction to realize a highest weight crystal [Kas90, Kas91] $B(\lambda)$ as a natural subset of $B(m\lambda)$ by dilating the crystal operators by m . Furthermore, Kashiwara gave necessary criteria for a generalization by realizing a $U_q(\mathfrak{g})$ -crystal inside of a $U_q(\widehat{\mathfrak{g}})$ -crystal via considering a diagram folding of type $\widehat{\mathfrak{g}}$ onto type \mathfrak{g} . This realization and the corresponding isomorphism is known as a virtual crystal and virtualization map (the latter is also known as a similarity map) respectively.

Virtual crystals have been used effectively to reduce problems into simply-laced types [Bak00, OSS03b, OSS03c, SS15a, SS15b, Scr15], where it is typically easier to prove certain properties. Most notably, there is a set of axioms, known as the Stembridge axioms [Ste03], which determines whether or not a crystal arises from a representation. In contrast, there are only limited partial results for type B_2 [DKK09, Ste07].

While there are numerous models for crystals, see, e.g., [GL05, Kam10, KN94, LP08, Lit95b, Nak03, SS15a], many of them have not had their behavior under virtualization studied. Virtualization of the Kashiwara–Nakashima tableau model was studied in [Bak00, OSS03b, OSS03c, SS15b], where the proofs were type-dependent and often involved tedious calculations. From computer experiments done by the second author using [Sag15], Nakajima monomials do not have nice behavior under virtualization. However, the situation is very different in other models. For rigged configurations, the virtualization map acts in a natural fashion [OSS03c, SS15a, SS15b]. Additionally, the virtualization map for the polyhedral realization [Nak99, NZ97], a semi-infinite tensor product of certain abstract crystals \mathcal{B}_i , is also well-behaved and is the setting which Kashiwara proved his criteria [Kas96] for similarity.

The goal of this note is to describe the virtualization map for the Littelmann path model [Lit95b], which is given by paths in the real weight space. We show that the virtualization map is the induced map on the weight spaces. This map is natural as it reflects the fact that the Littelmann path model is based upon the geometry of the root system. As a result, we give precise conditions for the existence of this type of virtualization map, recovering Kashiwara’s criteria. Moreover, as the alcove model [LP08] is a discrete version of the Littelmann path model and LS galleries [GL05] and MV polytopes [Kam10] are based on the root system geometry, we expect a similar natural virtualization maps for these models.

We remark that this work partially overlaps with the work of Naito and Sagaki on LS paths invariant under a diagram automorphism [NS01, NS05a, NS10]. In their case, they only consider

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the so-called scaling factors all equal to 1 and allow automorphisms where there may be adjacent vertices in an orbit (e.g., the middle edge in the natural type A_{2n} folding). Whereas in our case, we allow for arbitrary scaling factors and the vertices in the orbits of our automorphisms must not be adjacent.

There are a particular class of finite-dimensional affine crystals called Kirillov–Reshetikhin (KR) crystals [FOS09] that have many fascinating properties [Cha01, FOS10, Kle98, OSS03a]. Recently, tensor products of KR crystals of the form $\bigotimes_{i=1}^N B^{r_i,1}$ were constructed by Naito and Sagaki using projected level-zero LS paths in [NS03, NS06, NS08a, NS08b] and using quantum LS paths and the quantum alcove model in [LNS⁺14a, LNS⁺14b]. It was conjectured that KR crystals are also well-behaved under virtualization [OSS03c] and has been proven in a variety of special cases in a type-dependent fashion [OSS03c, OSS03b, SS15b]. As an application, we use the projected level-zero construction of Naito and Sagaki to give partial uniform results of the aforementioned conjecture.

This note is organized as follows. In Section 2, we give the necessary background on crystals, the Littelmann path model, level-zero crystals, and Kirillov–Reshetikhin crystals. In Section 3, we prove our main result, that the map on weight lattices naturally induces a virtualization map. In Section 4, we apply our main result to show special cases of the conjectural virtualization of KR crystals.

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2. BACKGROUND

Let \mathfrak{g} be a (symmetrizable) Kac–Moody algebra with index set I , (generalized) Cartan matrix $A = (A_{ij})_{i,j \in I}$, fundamental weights $\{\Lambda_i \mid i \in I\}$, simple roots $\{\alpha_i \mid i \in I\}$, simple coroots $\{\alpha_i^\vee \mid i \in I\}$, weight lattice P , coweight lattice P^\vee , root lattice Q , and coroot lattice Q^\vee . Let $U_q(\mathfrak{g})$ be the corresponding quantum group, $\mathfrak{h}_\mathbb{R}^* = \mathbb{R} \otimes_\mathbb{Z} P$, and $\mathfrak{h}_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Z} P^\vee$ be the corresponding dual space. Let $\langle \cdot, \cdot \rangle : \mathfrak{h}_\mathbb{R}^* \times \mathfrak{h}_\mathbb{R} \rightarrow \mathbb{R}$ denote the canonical pairing by evaluation, in particular $\langle \alpha_i^\vee, \alpha_j \rangle = A_{ij}$ and $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Let $\{s_i \mid i \in I\}$ denote the set of simple reflections on P , where $s_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i$. Let P^+ denote the positive weight lattice, i.e., all nonnegative linear combinations of the fundamental weights.

2.1. Crystals. An *abstract $U_q(\mathfrak{g})$ -crystal* is a non-empty set B along with maps

$$\begin{aligned} e_i, f_i &: B \rightarrow B \sqcup \{0\}, \\ \varepsilon_i, \varphi_i &: B \rightarrow \mathbb{Z} \sqcup \{-\infty\}, \\ \text{wt} &: B \rightarrow P, \end{aligned}$$

which satisfy the conditions:

- (1) $\varphi_i(b) = \varepsilon_i(b) + \langle \alpha_i^\vee, \text{wt}(b) \rangle$,
- (2) if $e_i b \in B$, then
 - (a) $\varepsilon_i(e_i b) = \varepsilon_i(b) - 1$,
 - (b) $\varphi(e_i b) = \varphi_i(b) + 1$,
 - (c) $\text{wt}(e_i b) = \text{wt}(b) + \alpha_i$,
- (3) if $f_i b \in B$, then
 - (a) $\varepsilon_i(f_i b) = \varepsilon_i(b) + 1$,
 - (b) $\varphi(f_i b) = \varphi_i(b) - 1$,
 - (c) $\text{wt}(f_i b) = \text{wt}(b) - \alpha_i$,
- (4) $f_i b = b'$ if and only if $b = e_i b'$ for $b' \in B$,
- (5) if $\varphi_i(b) = -\infty$, then $e_i b = f_i b = 0$,

for all $b \in B$ and $i \in I$. The maps e_i and f_i are known as the *crystal operators*.

We will only consider *regular crystals* in this note; that is every crystal in this note will satisfy

$$\varepsilon_i(b) = \max\{k \in \mathbb{Z} \mid e_i^k b \neq 0\},$$

$$\varphi_i(b) = \max\{k \in \mathbb{Z} \mid f_i^k b \neq 0\}.$$

Therefore the entire i -string through an element $b \in B$ can be given diagrammatically as

$$e_i^{\varepsilon_i(b)} b \xrightarrow{i} \cdots \xrightarrow{i} e_i b \xrightarrow{i} b \xrightarrow{i} f_i b \xrightarrow{i} \cdots \xrightarrow{i} f_i^{\varphi_i(b)} b.$$

Let B_1 and B_2 be abstract $U_q(\mathfrak{g})$ -crystals. A *crystal morphism* $\psi: B_1 \rightarrow B_2$ is a map $B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$ such that

- (1) $\psi(0) = 0$;
- (2) if $b \in B_1$ and $\psi(b) \in B_2$, then $\text{wt}(\psi(b)) = \text{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$ for all $i \in I$;
- (3) for $b, b' \in B_1$, $f_i b = b'$, and $\psi(b), \psi(b') \in B_2$, we have $\psi(f_i b) = f_i \psi(b)$ and $\psi(e_i b') = e_i \psi(b')$ for all $i \in I$.

2.2. Littelmann path model. Let $\pi, \pi': [0, 1] \rightarrow \mathfrak{h}_{\mathbb{R}}^*$, and define an equivalence relation \sim by saying $\pi \sim \pi'$ if there exists a piecewise-linear, nondecreasing, surjective, continuous function $\phi: [0, 1] \rightarrow [0, 1]$ such that $\pi = \pi' \circ \phi$. Let Π be the set of all piecewise-linear functions $\pi: [0, 1] \rightarrow \mathfrak{h}_{\mathbb{R}}^*$ such that $\pi(0) = 0$ modulo \sim . We call the elements of Π *paths*.

Let $\pi_1, \pi_2 \in \Pi$. Let $\pi = \pi_1 * \pi_2$ denote the concatenation of π_1 by π_2 :

$$\pi(t) := \begin{cases} \pi_1(2t) & 0 \leq t \leq 1/2, \\ \pi_1(1) + \pi_2(2t - 1) & 1/2 < t \leq 1. \end{cases}$$

For $\pi \in \Pi$, define $s_i(\pi)$ as the path given by $(s_i(\pi))(t) := s_i(\pi(t))$. Define the path π^\vee by $\pi^\vee(t) = \pi(1 - t) - \pi(1)$.

Define a function $H_{i,\pi}: [0, 1] \rightarrow \mathbb{R}$ by

$$t \mapsto \langle \alpha_i^\vee, \pi(t) \rangle,$$

and so we can express any path $\pi \in \Pi$ as

$$\pi(t) = \sum_{i \in I} H_{i,\pi}(t) \Lambda_i. \tag{2.1}$$

Let $m_i(\pi) := \min\{H_{i,\pi}(t) \mid t \in [0, 1]\}$ denote the minimal value of $H_{i,\pi}$.

We want to define $e_i^{(k)}$, where $k \in \mathbb{Z}_{>0}$. If $m_i(\pi) \leq -k$, then fix $t_1 \in [0, 1]$ minimal such that $H_{i,\pi}(t_1) = m_i(\pi)$ and let $t_0 \in [0, 1]$ maximal such that $H_{i,\pi}(t) \geq m_i(\pi) + k$ for $t \in [0, t_0]$. Choose $t_0 = t_{(0)} < t_{(1)} < \cdots < t_{(r)} = t_1$ such that either

- (1) $H_{i,\pi}(t_{(j-1)}) = H_{i,\pi}(t_{(j)})$ and $H_{i,\pi}(t) \geq H_{i,\pi}(t_{(j-1)})$ for $t \in [t_{(j-1)}, t_{(j)}]$;
- (2) or $H_{i,\pi}$ is strictly decreasing on $[t_{(j-1)}, t_{(j)}]$ and $H_{i,\pi}(t) \geq H_{i,\pi}(t_{(j)})$ for $t \leq t_{(j-1)}$.

Set $t_{(-1)} := 0$ and $t_{(r+1)} := 1$, and denote by $\pi_{(j)}$ the path defined by

$$\pi_{(j)}(t) := \pi(t_{(j-1)} + t(t_{(j)} - t_{(j-1)})) - \pi(t_{(j-1)})$$

for $i = 0, 1, \dots, r+1$.

Definition 2.1. Fix some $k \in \mathbb{Z}_{>0}$. If $m_i(\pi) > -k$, then $e_i^{(k)} \pi = 0$. Otherwise,

$$e_i^{(k)} \pi = \pi_{(0)} * \eta_{(1)} * \eta_{(2)} * \cdots * \eta_{(r)} * \pi_{(r+1)},$$

where $\eta_{(j)} = \pi_{(j)}$ if $H_{i,\pi}$ behaves as in (1) and $\eta_{(j)} = s_i(\pi_{(j)})$ if $H_{i,\pi}$ behaves as in (2) on $[t_{(j-1)}, t_{(j)}]$.

Next we want to define $f_i^{(k)}$, where $k \in \mathbb{Z}_{>0}$. Let $\bar{t}_0 \in [0, 1]$ be maximal such that $H_{i,\pi}(\bar{t}_0) = m_i(\pi)$. If $H_{i,\pi}(1) - m_i(\pi) \geq k$, then fix $\bar{t}_1 \in [\bar{t}_0, 1]$ minimal such that $H_{i,\pi}(t) \geq m_i(\pi) + k$ for $t \in [\bar{t}_1, 1]$. Choose $\bar{t}_0 = \bar{t}_{(0)} < \bar{t}_{(1)} < \dots < \bar{t}_{(r)} = \bar{t}_1$ such that either

- (1) $H_{i,\pi}(\bar{t}_{(j-1)}) = H_{i,\pi}(\bar{t}_{(j)})$ and $H_{i,\pi}(t) \geq H_{i,\pi}(\bar{t}_{(j-1)})$ for $t \in [\bar{t}_{(j-1)}, \bar{t}_{(j)}]$; or
- (2) $H_{i,\pi}$ is strictly increasing on $[\bar{t}_{(j-1)}, \bar{t}_{(j)}]$ and $H_{i,\pi}(t) \geq H_{i,\pi}(\bar{t}_{(j)})$ for $t \geq \bar{t}_{(j)}$.

Let $\bar{t}_{(-1)} := 0$ and $\bar{t}_{(r+1)} := 1$, and denote by $\bar{\pi}_{(j)}$ the path defined by

$$\bar{\pi}_{(j)}(t) := \pi(\bar{t}_{(j-1)} + t(\bar{t}_{(j)} - \bar{t}_{(j-1)})) - \pi(\bar{t}_{(j-1)})$$

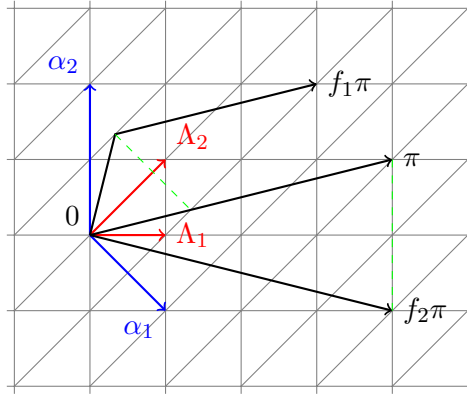
for $i = 0, 1, \dots, r+1$. It is clear that $\pi = \bar{\pi}_{(0)} * \bar{\pi}_{(1)} * \dots * \bar{\pi}_{(r+1)}$.

Definition 2.2. Fix some $k \in \mathbb{Z}_{>0}$. If $H_{i,\pi}(1) - m_i(\pi) < k$, then $f_i^{(k)}\pi = 0$. Otherwise,

$$f_i^{(k)}\pi = \bar{\pi}_{(0)} * \bar{\eta}_{(1)} * \bar{\eta}_{(2)} * \dots * \bar{\eta}_{(r)} * \bar{\pi}_{(r+1)},$$

where $\bar{\eta}_{(j)} = \bar{\pi}_{(j)}$ if $H_{i,\pi}$ behaves as in (1) and $\bar{\eta}_{(j)} = s_i(\bar{\pi}_{(j)})$ if $H_{i,\pi}$ behaves as in (2) on $[\bar{t}_{(j-1)}, \bar{t}_{(j)}]$.

Example 2.3. Consider \mathfrak{g} of type C_2 and the highest weight vector $\pi \in B(3\Lambda_1 + \Lambda_2)$. Thus we have



Remark 2.4. If $k = 1$, we will simply write e_i and f_i for $e_i^{(1)}$ and $f_i^{(1)}$ respectively.

Lemma 2.5 ([Kas96]). *We have*

$$e_i^{(k)} = e_i^k \quad \text{and} \quad f_i^{(k)} = f_i^k$$

for all $i \in I$ and $k \in \mathbb{Z}_{>0}$.

Note that

$$f_i^{(k)}(\pi) = (e_i^{(k)}(\pi^\vee))^\vee. \quad (2.2)$$

Fix a $\lambda \in P$. Let $B(\lambda)$ denote the closure under e_i and f_i , where $i \in I$, of the straight line path π from 0 to λ , where $\pi(t) = t\lambda$.

Theorem 2.6 ([Kas96, Lit95a, Lit95b]). *The set $B(\lambda)$ is a $U_q(\mathfrak{g})$ -crystal. Moreover, if $\lambda \in P^+$, then $B(\lambda)$ is the highest weight crystal of weight λ .*

We will also need the following fact.

Lemma 2.7 ([Lit95b, Lemma 4.5]). *For each $\pi \in B(\lambda)$ and $i \in I$, all local minimums of $H_{i,\pi}(t)$ and $H_{i,\pi}(1)$ are integers.*

2.3. Virtualization maps. Let A and \widehat{A} be Cartan matrices with indexing sets I and \widehat{I} respectively. Consider a diagram folding $\phi: \widehat{I} \rightarrow I$ with scaling factors $(\gamma_i)_{i \in I}$. Let $\Psi: \mathfrak{h}_{\mathbb{R}}^* \rightarrow \widehat{\mathfrak{h}}_{\mathbb{R}}^*$ be the map of the corresponding weight spaces given by

$$\Lambda_i \mapsto \sum_{j \in \phi^{-1}(i)} \gamma_j \widehat{\Lambda}_j. \quad (2.3)$$

Definition 2.8. Let \widehat{B} be a $U_q(\widehat{\mathfrak{g}})$ -crystal and $V \subseteq \widehat{B}$. Let ϕ and $(\gamma_a)_{a \in I}$ be the folding and the scaling factors. The *virtual crystal operators* (of type \mathfrak{g}) are defined as

$$e_i^v = \prod_{j \in \phi^{-1}(i)} \widehat{e}_j^{\gamma_i}, \quad (2.4a)$$

$$f_i^v = \prod_{j \in \phi^{-1}(i)} \widehat{f}_j^{\gamma_i}. \quad (2.4b)$$

A *virtual crystal* is a pair (V, \widehat{B}) such that V has a $U_q(\mathfrak{g})$ -crystal structure defined by

$$\begin{aligned} e_i &:= e_i^v & f_i &:= f_i^v, \\ \varepsilon_i &:= \gamma_i^{-1} \widehat{\varepsilon}_j & \varphi_i &:= \gamma_i^{-1} \widehat{\varphi}_j, \\ \text{wt} &:= \Psi^{-1} \circ \widehat{\text{wt}}. \end{aligned} \quad (2.5)$$

for any $j \in \phi^{-1}(i)$.

2.4. Level-zero and Kirillov–Reshetikhin crystals. In this section, we describe two classes of crystals of affine type that will be used in Section 4: level-zero crystals and Kirillov–Reshetikhin crystals. For this section, let \mathfrak{g} be of affine type.

For type \mathfrak{g} , let $0 \in I$ denote the special node, and let $I_0 := I \setminus \{0\}$ be the index set of the corresponding classical type \mathfrak{g}_0 . Let $\delta = \sum_{i \in I} a_i \alpha_i$ and $c = \sum_{i \in I} a_i^\vee \alpha_i^\vee$ denote the null root and the canonical central element of \mathfrak{g} respectively, where a_i and a_i^\vee are the Kac and dual Kac labels, respectively, as given in [Kac90, Table Aff. 1].

We say a weight $\lambda \in P$ is a *level-zero weight* if $\lambda(c) = 0$. A level-zero weight is *level-zero dominant* if $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all $i \in I_0$. The *level-zero fundamental weights* $\{\varpi_i \in P \mid i \in I_0\}$ are defined as

$$\varpi_i = \Lambda_i - a_i^\vee \Lambda_0. \quad (2.6)$$

Let $U'_q(\mathfrak{g}) := U_q([\mathfrak{g}, \mathfrak{g}])$ be the quantum group corresponding to the derived subalgebra of \mathfrak{g} . Define the *classical projection* as $\text{cl}: \mathfrak{h}_{\mathbb{R}}^* \rightarrow \mathfrak{h}_{\mathbb{R}}^*/\mathbb{R}\delta$ as the canonical projection. We identify the weight lattice of $U'_q(\mathfrak{g})$ with $P_{\text{cl}} := \{\text{cl}(\lambda) \mid \lambda \in P\}$.

We now recall an important class of finite-dimensional $U'_q(\mathfrak{g})$ -modules called *Kirillov–Reshetikhin (KR) modules* denoted by $W^{r,s}$, where $r \in I_0$ and $s \in \mathbb{Z}_{>0}$. KR modules are classified by their Drinfeld polynomials [CP95, CP98], and are the minimal affinizations of the highest weight module $V(s\overline{\Lambda}_r)$, where $\overline{\Lambda}_r$ is a fundamental weight of $U_q(\mathfrak{g}_0)$ [Cha01]. KR modules conjecturally admit a crystal basis [HKO⁺99, HKO⁺02], which has been shown to exist for all non-exceptional types in [OS08] and in certain exceptional types [KMOY07, LNS⁺14a, LNS⁺14b, Yam98]. The corresponding crystal to $W^{r,s}$ is known as a *Kirillov–Reshetikhin (KR) crystal* and is denoted by $B^{r,s}$. KR crystals are known to have deep connections with mathematical physics and are typically *perfect* [FOS10], a technical condition which affords many nice representation theoretic properties (see, e.g., [Bak00, HK02, KKM⁺92]).

KR crystals are also conjecturally well-behaved under the natural virtualization induced from the diagram folding ϕ given by the well-known natural embeddings of algebras [JM85]:

$$\begin{aligned}
C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)} &\hookrightarrow A_{2n-1}^{(1)}, \\
B_n^{(1)}, A_{2n-1}^{(2)} &\hookrightarrow D_{n+1}^{(1)}, \\
E_6^{(2)}, F_4^{(1)} &\hookrightarrow E_6^{(1)}, \\
G_2^{(1)}, D_4^{(3)} &\hookrightarrow D_4^{(1)}.
\end{aligned} \tag{2.7}$$

In this case, we define the scaling factors $(\gamma_i)_{i \in I}$ as follows.

- (1) Suppose the Dynkin diagram of \mathfrak{g} has a unique arrow.
 - (a) Suppose the arrow points towards the component of the special node 0. Then $\gamma_a = 1$ for all $a \in I$.
 - (b) Otherwise, γ_a is the order of ϕ for all a in the component of 0 after removing the arrow and $\gamma_a = 1$ in all other components.
- (2) Otherwise the Dynkin diagram of \mathfrak{g} has 2 arrows and is a folding of type $A_{2n-1}^{(1)}$. Then $\gamma_a = 1$ for all $1 \leq a \leq n-1$, and for $a \in \{0, n\}$, we have $\gamma_a = 2$ if the arrow points away from a and $\gamma_a = 1$ otherwise.

Conjecture 2.9. [OSS03c, Conj. 3.7] *The KR crystal $B^{r,s}$ of type \mathfrak{g} virtualizes into*

$$\widehat{B}^{r,s} = \begin{cases} B^{n,s} \otimes B^{n,s} & \text{if } \mathfrak{g} = A_{2n}^{(2)}, A_{2n}^{(2)\dagger} \text{ and } a = n, \\ \bigotimes_{b \in \phi^{-1}(a)} B^{b, \gamma_a s} & \text{otherwise.} \end{cases}$$

Conjecture 2.9 was shown for $B^{r,1}$ in types $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, and $C_n^{(1)}$ in [OSS03b] and types $E_6^{(2)}$, $F_4^{(1)}$ (except $r = 2$), $G_2^{(1)}$, and $D_4^{(3)}$ in [SS15b], $B^{1,s}$ for all non-exceptional types [OSS03c], and $B^{r,s}$ for $r < n$ in types $B_n^{(1)}$ and $A_{2n-1}^{(2)}$ [SS15b].

We can extend cl to paths in a natural way by $\text{cl}(\pi)(t) = \text{cl}(\pi(t))$. Now we can define the set of *projected level-zero paths* $B(\lambda)_{\text{cl}} := \{\text{cl}(\pi) \mid \pi \in B(\lambda)\}$. In [NS05b], it was shown that $B(\lambda)_{\text{cl}}$ has a $U'_q(\mathfrak{g})$ -crystal structure inherited from the $U_q(\mathfrak{g})$ -crystal structure on $B(\lambda)$. We have the following key result that follows from [NS03, NS06, NS08a, NS08b].

Theorem 2.10. *Let \mathfrak{g} be of affine type. Let $\lambda = \sum_{i \in I_0} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\geq 0}$, is a dominant level-zero weight. Then there exists a canonical $U'_q(\mathfrak{g})$ -crystal isomorphism*

$$B(\lambda)_{\text{cl}} \rightarrow \bigotimes_{i \in I_0} (B^{i,1})^{\otimes m_i}.$$

3. MAIN RESULTS

In this section we prove our main result.

Theorem 3.1. *Let ϕ be a diagram folding. The induced map $\Psi: P \rightarrow \widehat{P}$ given by Equation (2.3) induces a virtualization map $\widetilde{\Psi}: B(\lambda) \rightarrow B(\Psi(\lambda))$ given by*

$$\widetilde{\Psi}(\pi)(t) = \sum_{i \in I} H_{i,\pi}(t) \Psi(\Lambda_i)$$

if and only if the following properties hold:

- (I) *For all $j \neq j' \in \phi^{-1}(i)$, we have $\widehat{A}_{j,j'} = 0$ (i.e., in the Dynkin diagram of $\widehat{\mathfrak{g}}$, the vertices of j and j' are not adjacent).*

(II) We have

$$\Psi(\alpha_i) = \sum_{j \in \phi^{-1}(i)} \gamma_j \hat{\alpha}_j. \quad (3.1)$$

Proof. Fix some $\pi \in B(\lambda)$, and set

$$\hat{\pi}(t) := \tilde{\Psi}(\pi)(t) = \sum_{i \in I} H_{i,\pi}(t) \sum_{j \in \phi^{-1}(i)} \gamma_j \hat{\Lambda}_j.$$

It is clear that $\tilde{\Psi}(\pi^\vee) = (\tilde{\Psi}(\pi))^\vee$. So $\tilde{\Psi}$ is a virtualization map if and only if $\tilde{\Psi}(e_i \pi) = e_i^v \tilde{\Psi}(\pi)$ by Equation (2.2). Note that property (I) holds if and only if the virtual crystal operators are well-defined.

Fix some $i \in I$. We first consider the case when $e_i \pi = 0$. Thus we have $m_i(\pi) > -1$. By Lemma 2.7, we can assume $m_i(\pi) \geq 0$. Next, take any $k \in \phi^{-1}(i)$. Then

$$\begin{aligned} m_k(\hat{\pi}) &= \min\{H_{k,\hat{\pi}}(t) \mid t \in [0, 1]\} \\ &= \min\{\langle \hat{\alpha}_k^\vee, \hat{\pi}(t) \rangle \mid t \in [0, 1]\} \\ &= \min\{\gamma_k H_{i,\pi}(t) \mid t \in [0, 1]\} \\ &= \gamma_k m_i(\pi) \geq 0. \end{aligned}$$

Therefore $\hat{e}_k \hat{\pi} = 0$ for any $k \in \phi^{-1}(i)$.

Now we consider the case when $e_i \pi \neq 0$. Thus, for any $j \in \phi^{-1}(i)$, we have

$$H_{j,\hat{\pi}}(t) = \langle \hat{\alpha}_j^\vee, \tilde{\Psi}(\pi)(t) \rangle = \gamma_j H_{i,\pi}(t).$$

Fix some division of π into subpaths

$$e_i^{(k)} \pi = \pi_{(0)} * \eta_{(1)} * \eta_{(2)} * \cdots * \eta_{(r)} * \pi_{(r+1)}$$

as in Definition 2.1. Note that

$$\tilde{\Psi}(\pi) = \tilde{\Psi}(\pi_{(0)}) * \tilde{\Psi}(\pi_{(1)}) * \tilde{\Psi}(\pi_{(2)}) * \cdots * \tilde{\Psi}(\pi_{(r)}) * \tilde{\Psi}(\pi_{(r+1)})$$

since conditions (1) and (2) for the subdivision of π still hold when $H_{i,\pi}$ is scaled by a positive constant. Thus we have

$$\tilde{\Psi}(e_i \pi) = \tilde{\Psi}(\pi_{(0)}) * \tilde{\Psi}(\eta_{(1)}) * \tilde{\Psi}(\eta_{(2)}) * \cdots * \tilde{\Psi}(\eta_{(r)}) * \tilde{\Psi}(\pi_{(r+1)}),$$

and to show

$$e_i^v \tilde{\Psi}(\pi) = \hat{\pi}_{(0)} * \hat{\pi}_{(1)} * \hat{\pi}_{(2)} * \cdots * \hat{\pi}_{(r)} * \hat{\pi}_{(r+1)} = \tilde{\Psi}(e_i \pi),$$

it is sufficient to show that $\hat{\eta}_{(k)} = \tilde{\Psi}(\eta_{(k)})$ for all $1 \leq k \leq r$ since $\hat{\pi}_{(0)} = \tilde{\Psi}(\pi_{(0)})$ and $\hat{\pi}_{(r+1)} = \tilde{\Psi}(\pi_{(r+1)})$.

Fix some $1 \leq k \leq r$. If $\eta_{(k)} = \pi_{(k)}$, then $H_{i,\pi}(t_{(k-1)}) = H_{i,\pi}(t_{(k)})$ and $H_{i,\pi}(t) \geq H_{i,\pi}(t_{(k-1)})$ for $t \in [t_{(k-1)}, t_{(k)}]$, i.e., condition (1) is satisfied. So for any $j \in \phi^{-1}(i)$ and $t \in [t_{(k-1)}, t_{(k)}]$, we obtain

$$H_{j,\hat{\pi}}(t_{(k)}) = \gamma_j H_{i,\pi}(t_{(k)}) = \gamma_j H_{i,\pi}(\hat{t}_{(k-1)}) = H_{j,\hat{\pi}}(t_{(k-1)}),$$

and

$$H_{j,\hat{\pi}}(t) = \gamma_j H_{i,\pi}(t) \geq \gamma_j H_{i,\pi}(t_{(k-1)}) = H_{j,\hat{\pi}}(t_{(k-1)})$$

since γ_i is positive. Therefore $\hat{\eta}_{(k)} = \tilde{\Psi}(\pi_{(k)}) = \tilde{\Psi}(\eta_{(k)})$.

Now suppose $\eta_{(k)} = s_i \pi_{(k)}$, which implies $H_{i,\pi}$ is strictly decreasing on $[t_{(k-1)}, t_{(k)}]$ and $H_{i,\pi}(t) \geq H_{i,\pi}(t_{(k)})$ for $t \leq t_{(k-1)}$, i.e., condition (2) is satisfied. Then

$$\eta_{(k)}(t) = (s_i \pi_{(k)})(t) = s_i(\pi_{(k)}(t)) = \pi_{(k)}(t) - \langle \alpha_i^\vee, \pi_{(k)}(t) \rangle \alpha_i = \pi_{(k)}(t) - H_{i,\pi}(t) \alpha_i,$$

and

$$\tilde{\Psi}(\eta_{(k)})(t) = \Psi(\pi_{(k)}(t) - H_{i,\pi}(t)\alpha_i) = \Psi(\pi_{(k)}(t)) - H_{i,\pi}(t)\Psi(\alpha_i).$$

We note that from the definition of the crystal operators and Lemma 2.5, it is sufficient to consider $s_i^v = \prod_{j \in \phi^{-1}(i)} \hat{s}_j$ for the action of e_i^v . Thus we have

$$\begin{aligned} s_i^v(\Psi(\pi_{(k)}(t))) &= \prod_{j \in \phi^{-1}(i)} \hat{s}_j(\Psi(\pi_{(k)}(t))) \\ &= \prod_{j \in \phi^{-1}(i)} \hat{s}_j \left(\sum_{l \in I} H_{l,\pi}(t) \sum_{p \in \phi^{-1}(l)} \gamma_p \hat{\Lambda}_p \right) \\ &= \Psi(\pi_{(k)}(t)) - \sum_{j \in \phi^{-1}(i)} \left\langle \hat{\alpha}_j^\vee, \sum_{l \in I} H_{l,\pi}(t) \sum_{p \in \phi^{-1}(l)} \gamma_p \hat{\Lambda}_p \right\rangle \hat{\alpha}_j \\ &= \Psi(\pi_{(k)}(t)) - H_{i,\pi}(t) \sum_{j \in \phi^{-1}(i)} \gamma_j \hat{\alpha}_j. \end{aligned}$$

Hence $s_i^v(\tilde{\Psi}(\pi_{(k)})) = \tilde{\Psi}(\eta_{(k)})$ if and only if $\Psi(\alpha_i) = \sum_{j \in \phi^{-1}(i)} \hat{\alpha}_j$. Therefore $\tilde{\Psi}$ is a virtualization map if and only if Equation (3.1) holds. \square

Example 3.2. Consider the diagram folding from type A_3 to type C_2 , where $\phi^{-1}(1) = \{1, 3\}$ and $\phi^{-1}(2) = \{2\}$ and $\gamma_i = i$. The map Ψ is given by

$$\begin{aligned} \Lambda_1 &\mapsto \hat{\Lambda}_1 + \hat{\Lambda}_3, \\ \Lambda_2 &\mapsto 2\hat{\Lambda}_2. \end{aligned}$$

Since $\alpha_i = \sum_{i' \in I} A_{i',i} \Lambda_{i'}$, we have

$$\begin{aligned} \hat{\alpha}_1 + \hat{\alpha}_3 &= (2\hat{\Lambda}_1 - \hat{\Lambda}_2) + (2\hat{\Lambda}_3 - \hat{\Lambda}_2) = 2(\hat{\Lambda}_1 + \hat{\Lambda}_3) - 2\Lambda_2 = 2\Psi(\Lambda_1) - \Psi(\Lambda_2) = \Psi(\alpha_1), \\ 2\hat{\alpha}_2 &= 2(2\hat{\Lambda}_2 - \hat{\Lambda}_1 - \hat{\Lambda}_3) = 2(2\hat{\Lambda}_2) - 2(\hat{\Lambda}_1 + \hat{\Lambda}_3) = 2\Psi(\Lambda_2) - 2\Psi(\Lambda_1) = \Psi(\alpha_2). \end{aligned}$$

Therefore, the map $\tilde{\Psi}$ is a virtualization map by Theorem 3.1. In particular, if we consider the highest weight element $\pi(t) = 3t\Lambda_1 + t\Lambda_2 \in B(3\Lambda_1 + \Lambda_2)$, then

$$\begin{aligned} (f_2(\pi))(t) &= 5t\Lambda_1 - t\Lambda_2, \\ \tilde{\Psi}(\pi)(t) &= 3t\Lambda_1 + 2t\Lambda_2 + 3t\Lambda_3, \\ \tilde{\Psi}(f_2\pi)(t) &= (\hat{f}_2^2 \tilde{\Psi}(\pi))(t) = 5t\Lambda_1 - 2t\Lambda_2 + 5t\Lambda_3. \end{aligned}$$

4. APPLICATIONS

In this section, we present an application of Theorem 3.1 to KR crystals.

Proposition 4.1. *Let \mathfrak{g} be of affine type. Let $\tilde{\Psi}$ be the virtualization map induced from the diagram folding ϕ given in Section 2.4. Then there exists a $U'_q(\mathfrak{g})$ -crystal virtualization map $\tilde{\Psi}_{\text{cl}}$ such that*

the diagram

$$\begin{array}{ccc}
B(\lambda) & \xrightarrow{\tilde{\Psi}} & B(\Psi(\lambda)) \\
\downarrow \text{cl} & & \downarrow \text{cl} \\
B(\lambda)_{\text{cl}} & \xrightarrow{\tilde{\Psi}_{\text{cl}}} & B(\Psi(\lambda))_{\text{cl}}
\end{array}$$

commutes.

Proof. This can be seen from the fact that $\Psi(\delta) = a_0 \gamma_0 \hat{\delta}$. \square

Theorem 4.2. Let \mathfrak{g} be of affine type. Suppose $r \in I$ is such that $\gamma_r = 1$ or \mathfrak{g} is of type $A_{2n}^{(2)}$, $A_{2n}^{(2)\dagger}$. Then Conjecture 2.9 holds for $s = 1$.

Proof. From Theorem 3.1, there exists a virtualization map $\tilde{\Psi}: B(\varpi_r) \rightarrow B(\Psi(\varpi_r))$ from the diagram folding ϕ given in Section 2.4. From Proposition 4.1 and Theorem 2.10, the result follows. \square

APPENDIX A. EXAMPLES WITH SAGE

We give some examples using Sage [Sag15].

```

sage: LaC = RootSystem(['C', 2]).weight_space().fundamental_weights()
sage: LaA = RootSystem(['A', 3]).weight_space().fundamental_weights()
sage: BA = crystals.LSPaths(LaA[1]+LaA[3])
sage: BC = crystals.LSPaths(LaC[1])
sage: list(BC)
[(Lambda[1],),
 (-Lambda[1] + Lambda[2],),
 (Lambda[1] - Lambda[2],),
 (-Lambda[1],)]
sage: mg = BA.module_generators[0]; mg
(Lambda[1] + Lambda[3],)
sage: x1 = mg.f_string([1, 3]); x1
(-Lambda[1] + 2*Lambda[2] - Lambda[3],)
sage: x2 = x1.f_string([2, 2]); x2
(Lambda[1] - 2*Lambda[2] + Lambda[3],)
sage: x3 = x2.f_string([1, 3]); x3
(-Lambda[1] - Lambda[3],)

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